Exam Graph Theory

20 January 2025, 18.15–20.15

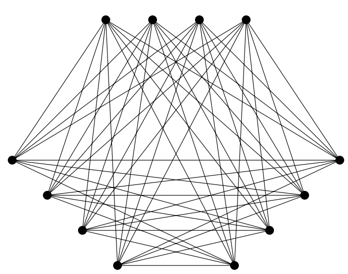
- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Make sure to state clearly any results from the lecture notes you are using.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet**. This is worth 10 points (out of a total of 100). Nota bene: by a 'sheet' we mean a folded booklet with the university logo on the front.

Exercise 1 (20 points)
Determine the result of the Gale–Shapley algorithm (in which the students propose projects to the professors) on the situation below. Make sure to clearly indicate, for each step of the algorithm, what actions are taken by the algorithm.

	professors' preferences
p_1 :	$s_1 > s_2 > s_4 > s_3$
p_2 :	$s_4 > s_3 > s_1 > s_2$
p_3 :	$s_3 > s_4 > s_1 > s_2$
p_4 :	$s_4 > s_2 > s_3 > s_1$

	students' preferences
s_1 :	$p_1 > p_2 > p_3 > p_4$
s_2 :	$p_3 > p_4 > p_1 > p_2$
s_3 :	$p_1 > p_3 > p_2 > p_4$
s_4 :	$p_3 > p_2 > p_1 > p_4$

Exercise 2 (20 points)
Let G be the complete tripartite graph $K_{4,4,4}$ (see picture). Show that $\chi(G) < \chi_\ell(G)$.



Exercise 3 (20 points)
Let G be a graph in which any two odd cycles share a common vertex. Prove that $\chi(G) \leq 5$.

Hint: Consider removing a well-chosen cycle from G.

Exercise 4 (30 points)
(a) Give an example of a tree with 4 vertices that does not have a perfect matching and show that it does not have a perfect matching.
(b) Prove that any tree has at most one perfect matching.

Bonus: You will get 10 extra points if you provide a second solution for 4(b) that uses a different idea than your first proof.

(The end)